INVESTIGATION OF AXISYMMETRIC TRANSONIC FLOW BY MEANS OF A SPECIAL HODOGRAPH PLANE

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A special hodograph plane is introduced for the study of axisymmetric transonic flow. Investigation of mappings in this plane permits the generalization of a number of known relations for plane transonic flow.

The equations of axisymmetric transonic flow have the form [1]

$$uu_x = v_y + v / y, \quad v_x = u_y \quad (u = (k+1)(\lambda - 1) +, \ \ldots, \ v = (k+1)\beta + \ldots)$$

Here λ is the speed coefficient, β the angle of inclination of the velocity vector with the axis of symmetry, and x, y are a Cartesian coordinate system in the physical plane (the x-axis coinciding with the axis of symmetry, and $y = |y|$). Setting $w = vy$, we reduce this system to a form that is homogeneous (with respect to first derivatives):

$$yuu_x = w_y, \qquad w_x = yu_y \tag{1}$$

We introduce the special uw hodograph plane. This plane is obtained by dilatation at each point of the uv hodograph plane by y times in the direction of the v-axis. Let the x and u axes be directed horizontally to the right, and the w and y axes vertically upward.

1. The Jacobian of the mapping in the uw plane is transformed with the aid of (1) to the form

$$J = \frac{\partial(u, w)}{\partial(x, y)} = u_x w_y - u_y w_x = y(uu_x^2 - u_y^2) \tag{2}$$

Since $J \leqslant 0$ for $u \leqslant 0$, the mapping of the region of subsonic speed in the uw plane is locally one-sheeted (J vanishes only at isolated points). In addition, the orientation of corresponding contours is opposite in the xy and uw planes. Consequently there is a generalization of the "law of monotonicity" of the velocity vector on the sonic line, established in [2] for plane potential flow, which is stated below.

Displacement along the sonic line, with the region of subsonic speeds remaining on the left, corresponds to monotonic decrease of w.

If y increases monotonically for such a displacement, then v also decreases, that is, the velocity vector rotates monotonically clockwise. The latter is true also in the framework of the exact equations of axisymmetric potential flow for the part of the sonic line where $\beta \geqslant 0$ and where the stream function ψ increases for the indicated direction of traversal. In fact, the equations of motion have the form where ($\partial / \partial S_1$ and $\partial / \partial S_2$ are derivatives in the streamline direction and normal to it.

$$(M^2 - 1)\frac{\partial \ln \lambda}{\partial S_1} = \frac{\partial \beta}{\partial S_2} + \frac{\sin \beta}{y}, \qquad \frac{\partial \beta}{\partial S_1} = \frac{\partial \ln \lambda}{\partial S_2} \tag{3}$$

For $\lambda = 1$ we obtain

$$\frac{\partial(\ln \lambda, \beta)}{\partial(x, y)} = \frac{\partial(\ln \lambda, \beta)}{\partial(S_1, S_2)} = -\frac{\sin \beta}{y}\frac{\partial \ln \lambda}{\partial S_1} - \left(\frac{\partial \beta}{\partial S_1}\right)^2 \leqslant 0 \text{ for } \beta \geqslant 0, \ \frac{\partial \lambda}{\partial S_1} \geqslant 0$$

For the chosen direction of traversal of the sonic line, the condition $d\lambda / dS_1 \geqslant 0$ is equivalent to the condition of monotonic increase of ψ.

The system (1) is transformed in the uw plane to the form

$$yuy_w = x_u, \qquad y_u = yx_w \tag{4}$$

The Jacobian $j = J^{-1}$ can, with the use of this equation, be written in the form

$$j = x_u y_w - x_w y_u = y(uy_w^2 - x_w^2)$$

In the subsonic region j can vanish only at isolated points.

2. The equations of the characteristics in the xy and uw planes have the form

$$\left(\frac{dy}{dx}\right)_{1,2} = \pm \frac{1}{\sqrt{u}}, \qquad \left(\frac{dw}{du}\right)_{1,2} = \pm y\sqrt{u} \tag{5}$$

From the first equation it follows that in a flow described by the system (1) the direction of the sonic line is characteristic at points K of verticality of the sonic line; in these points $J = 0$.

It follows from (5) that at corresponding points of the x, $\ln y$ and uw planes there is mutual orthogonality of the characteristics of opposite families.

If we denote derivatives along the characteristic directions in the xy and uw planes by

$$\frac{\partial}{\partial s_{1,2}} = \pm \sqrt{u}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \sigma_{1,2}} = \frac{\partial}{\partial u} \pm y\sqrt{u}\frac{\partial}{\partial w}$$

then the expressions for J and j transform to

$$J = y\frac{\partial u}{\partial s_1}\frac{\partial u}{\partial s_2} = -\frac{1}{uy}\frac{\partial w}{\partial s_1}\frac{\partial w}{\partial s_2}, \qquad j = uy\frac{\partial x}{\partial \sigma_1}\frac{\partial x}{\partial \sigma_2} = -\frac{1}{2}\frac{\partial y}{\partial \sigma_1}\frac{\partial y}{\partial \sigma_2} \tag{6}$$

3. Using the second equation (6) it is easily proved, in analogy to the case of plane potential flow [3], that the image of the vertex of a convex angle in the uw plane is a characteristic.

For flow past a convex angular point the velocity vector changes continuously, therefore its image is a continuous curve. The velocity at such a point is multi-valued; consequently $j = 0$ there, that is, at least one of the derivatives $\partial y / \partial \sigma_1$, $\partial y / \partial \sigma_2$ is equal to zero. If the image of the angular point is not a characteristic, the solution of the Cauchy problem with the conditions $y = \text{const}$, $\partial y / \partial \sigma = 0$ gives $y = \text{const}$ on all characteristics issuing from the angular point, which is impossible.

4. A line across which the Jacobian J in the xy plane changes sign is called a branch line; it is a folding edge of the image in the hodograph plane. In connection with the nonlinearity of the system (4) a branch line is not in general a characteristic. An exception is possible only when a discontinuity in the first derivatives of the velocity components propagates along the characteristic.

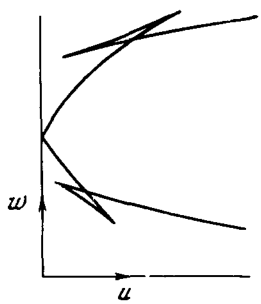

Fig. 1

In a sufficiently small neighborhood of a branch line the characteristics in the uw plane are located on one side of the folding edge. In view of the continuity of the tangent to a characteristic in the region of continuity of the velocity vector field (5) we obtain that in the general case a branch line in the uw plane consists of segments, each of which is an envelope of characteristics of one family and the locus of cusps of the other family. At a branch line there is a change in sign of derivatives of u and w in the direction of a characteristic of that family whose image in the plane is a cusp; the curvature of this characteristic changes sign in the physical plane.

5. From the theorem of existence of a solution of the first equation (5) it follows that in a region of continuous supersonic flow y is a monotonic function of the arc length along a characteristic. We transform the uw plane into the tw plane, where $t = (^2/_3)u^{3/2}$. The equation of the characteristics in the tw plane is

$$(dw \,/\, dt)_{1,2} = \pm\, y \tag{7}$$

so that the angle of inclination of a characteristic in the tw plane is a monotonic function of arc length. Consequently in the tw plane on each segment of a characteristic that does not contain a cusp the curvature of the characteristic has a constant sign; segments of a characteristic that adjoin a cusp are convex to each other (Fig. 1).

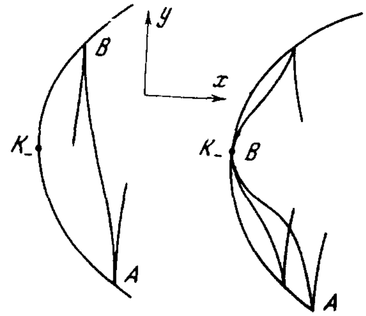

Fig. 2

6. We consider a characteristic emanating from an arbitrary point O of the sonic line. If y decreases for a displacement along it from the sonic line, then $w > w_0$ on the characteristic of the first family and $w < w_0$ on the characteristic of the second family.

It is sufficient to give the proof for the characteristic of the first family. We condider the case when there are cusps on the characteristic in the tw plane, since otherwise the proof is trivial. We divide the characteristic into segments between cusps, and number them outward from the sonic line. On the first segment we have $w > w_0$. The second segment lies not lower than the tangent to the first segment at its right-most point. Therefore on the second segment we also have $w > w_0$, and so on.

7. We will call a point K of verticality of the sonic line the point K_+(or K_-) if the sonic line at this point is convex on the side of the supersonic (or subsonic) region.

We show that within the flow region there exists no point K_-. We assume that a point K_- exists. We consider a point A of the sonic line at which $y_A < y_{K_-}$, and trace from it the characteristic AB of the second family. If point A is sufficiently close to point K_-, this characteristic again approaches the sonic line and intersects it, either at point K_- or at some point B where $y_B > y_{K_-}$ (Fig. 2) (*).

It follows from Sect. 6 that $w_A < w_B$. On the other hand, from the law of monotonicity of w on the sonic line from Sect. 1 we obtain the reverse inequality $w_B < w_A$. Thus the assumption of the existence of a point K_- has led to a contradiction. As a consequence we obtain from this that no more than one point K_+ can exist on the sonic line.

8. We determine the value of the angle of inclination of the solnic line at the body.

We will call the contour of the body whose equation is $y = (c_1 x + c_2)^{1/2}$ the w-line. If the curvature of the contour at any point is greater (less) than the curvature of the

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*) Point K_- exists for rotational flow. Then both cases of the arrangement of characteristics shown in Fig. 2 are possible [4].

w-line contour, we will call it w-convex (w-concave). The image of the w-line contour in the uw plane lies on the straight line w = const.

It follows from Eq. (1) that at the sonic point on the contour

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial x} = \frac{dw}{dx}$$

so that $u_y < 0$ at the sonic point of a w-convex contour and $u_y > 0$ at the sonic point of a w-concave contour.

Hence follows the rule: For flow past a w-convex (w-concave) contour the tangent to the sonic line at the contour is obtained by counterclockwise (clockwise) rotation through an acute angle of the direction of the x-axis that corresponds to increasing velocity.

We remark that in the framework of the full equations (3) another relation holds, which is asymptotically equivalent (for u, $v \to 0$) and differs from this in that the words " w-convex" and " x-axis" are replaced by the words "convex" and "velocity vector".

9. We determine the value of the angle of inclination of the sonic line at the sonic point of the shock wave that arises in uniform supersonic flow ahead of the body.

We denote by δ the acute angle between the tangent to the shock wave and the y-axis, with $\delta > 0$ if the angle is measured counterclockwise from the tangent to the shock wave. The relations at the shock wave in the transonic approximation are expressed in the form

$$u = 2\delta^2 - u_\infty, \qquad w = 2\delta y\,(u_\infty - \delta^2)$$

where u_∞ is the speed of the free stream.

Adjoining to Eqs. (1) the expressions for the derivatives of u and w in the direction of the shock wave, simplified in the transonic speed range,

$$u_x\delta + u_y = \frac{du}{dy} = \frac{du}{d\delta}\frac{d\delta}{dy} = 4\delta\delta'$$

$$w_x\delta + w_y = \frac{dw}{dy} = \frac{\partial w}{\partial \delta}\frac{d\delta}{dy} + \frac{\partial w}{\partial y} = 2\delta\,(u_\infty - \delta^2) + 2y\,(u_\infty - 3\delta^2)\,\delta'$$

we obtain a system for the determination of the derivatives u_x, u_y, w_x, w_y in the case when the curvature of the shock wave is bounded. (In the case of infinite curvature of the shock wave the singularity is the same as in plane flow).

At the sonic point of the shock wave

$$u_y = \frac{2\delta}{y}(\delta - y\delta'), \qquad u_x = \frac{2}{y}(3y\delta' - \delta)$$

so that we obtain for the angle of inclination γ of the sonic line to the y-axis

$$\operatorname{tg}\gamma = -\frac{u_y}{u_x} = \delta\,\frac{y\delta' - \delta}{3y\delta' - \delta}, \qquad \delta' = \frac{d\delta}{dy}$$

Table 1

N	1	2	3	4	5
d	$1<d<3$	$0<d<1$	$d<0$	$d>3$	$d<0$
δ	+	+	+	−	−
δ'	+	+	−	−	+
u_x	+	+	−	+	+
u_y	+	−	+	+	+
γ	−	+	+	−	−

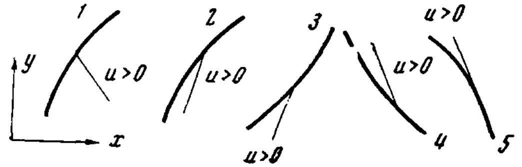

Fig. 3

Analyzing the signs of γ, u_x, u_y and δ', we find that there can exist only the cases shown in Table 1. The first row of the Table gives the case number, the second row the range of $d = \delta / y\delta'$, and the remaining rows the signs of the quantities δ, δ', u_x, u_y, γ. The cases are shown in Fig. 3 by the corresponding numbers (where the heavy line is the shock wave and the thin one the the sonic line).

The results of this Sect. agree asymptotically (for u, $w \to 0$) with the analogous relations obtained in [5] for the exact equations.

10. The results of Sects. 7–9 allow a classification to be established of the minimal regions of influence of the mixed flow for flow past bodies of various shapes with detached shock waves. For example, for flow past a w-convex body located on the axis of symmetry, only two types of minimal region of influence can exist (Fig. 4): with a point K_+ (for $1 < d < 3$) and without one (for $0 < d < 1$).

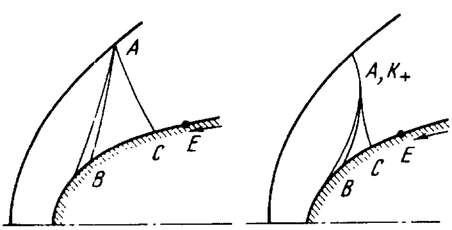

Fig. 4

11. The property of Sect. 6 permits generalization to the case of axisymmetric transonic flow of the result, established in [6] for plane flow, regarding the destruction by a specific deformation of the body of continuous supersonic flow in the characteristic triangle ABC (Fig. 4) adjacent to the minimal region of influence.

We assume the existence of a w-convex body such that in a uniform supersonic stream with detached shock wave there exists continuous supersonic flow in the triangle ABC (point A being either the sonic point on the shock wave or the point K_+ of the sonic line. In other words, existence is assumed "in the large" of a continuous solution of "Problem 3" ([7], p. 56) for a given distribution of velocity on the characteristic AB and the condition of no flow through the wall BC. In this case the bounds $w_B > w_A > w_C$ follow from Sect. 6. Therefore on the segment BC of the contour there exists a point D at which $w_D = w_A$.

We subject the body to a continuous deformation, changing its contour downstream of some point E of the tangent to the contour at this point and thus shifting point E upstream to point D. With the location of point E coinciding with point D, continuous supersonic flow in the triangle ABC will no longer exist because of violation of the bound $w_D > w_C$. Thus for point E sufficiently close to point D, Problem 3 in the triangle ABC will no longer have a solution in the large. This means that there appears either a local zone of subsonic speeds, or a shock wave.

12. We consider the subregion of supersonic speeds that is contained in the minimal region of influence of the mixed flow. This subregion is covered by characteristics of both families that originate form the sonic line; following [2] we will call it zone 1. Zone 1 may be of two types: in zone 1a the quantity y decreases with displacement along the characteristic in the direction of the sonic line, in zone 1b it increases.

We prove that the mapping of zone 1a into the tw plane is single-sheeted.

For a multiple-sheeted mapping (having in mind local single-sheetedness) a branch line appears, which in the tw plane is an envelope of characteristics of one family (Sect. 4). We show that characteristics drawn on the image of zone 1a do not intersect (and hence have no envelope).

We consider the characteristic $w_0 = w_0(t_0)$ in the tw plane that comes out of an arbitrary point of the sonic line; let it be a characteristic of the first family. Through each of its points we pass a sraight line tangent to the characteristic of the second family; in view of the smoothness of the characteristic $w_0 = w_0(t_0)$, these lines will not have an envelope in a sufficiently small neighborhood of it. Without loss of generality we may assume that the segment of the original characteristic under consideration does not contain a cusp.

We assign on each straight line a direction field parallel to the tangent to the characteristic $w_0 = w_0(t_0)$ at the point of intersection with this straight line. For determining in the vicinity of the original characteristic the integral curves $W = W(t)$ of this field we obtain the system

$$\frac{dW(t(t_0))}{dt} = \frac{dw_0(t_0)}{dt_0}, \qquad W(t) - w_0(t_0) = -(t - t_0)\frac{dw_0(t_0)}{dt_0}$$

Differentiating the second equation with respect to t_0 and eliminating dW / dt gives

$$2\frac{dw_0}{dt_0}\left(\frac{dt}{dt_0} - 1\right) = -\frac{d^2w_0}{dt_0^2}(t - t_0)$$

Hence

$$|t - t_0| = C\left|\frac{dw_0}{dt_0}\right|^{-1/2} = \frac{C}{\sqrt{y}}$$

If a segment of the characteristic $w_0 = w_0(t_0)$ adjoins the sonic line and does not contain a cusp, $y(t_0)$ is a decreasing function, and the auxiliary curve emitted from the line $t = 0$ does not intersect the characteristic $w_0 = w_0(t)$ for $C \neq 0$.

We suppose that in zone 1a there exists a branch line – an envelope of characteristics of the first family. We draw the segment of the characteristic $w_0 = w_0(t_0)$ from the line $t = 0$ to its intersection with the branch line at point O. In the vicinity of point O two cases may occur: in the first, the characteristic lies below the envelope; in the second, on the contrary, the characteristic lies above the envelope.

To begin with, we consider the first case (Fig. 5). In the strip $0 < t < t_0$ we draw the auxiliary curve $W = W(t)$ above the characteristic $w_0 = w_0(t_0)$. We denote by the numeral *1* the intersection of the curve $W = W(t)$ with the branch line; this point exists since $|t - t_0| > 0$. Because the curve $W = W(t)$ is sufficiently close to the characteristic $w_0 = w_0(t_0)$ we may assume that in the vicinity of point *1* the branch line *1* – 0 is located in the sector between the rays of the line $W(t)$ and the straight line $w = w_1$. on which $t > t_1$.

If we denote the angle of inclination of the characteristic of the first family at point *1* by $(dw / dt)_1$, then this means that

$$0 < \left(\frac{dw}{dt}\right)_1 \leqslant \left(\frac{dW}{dt}\right)_1 \tag{8}$$

We draw the segment *1—2* of the characteristic from point *1* to its intersection with the characteristic $w_0 = w_0(t_0)$ at point *2*. Because points *1* and 0 are sufficiently close, we may assume that the entire segment *1—2* lies on the same sheet of the Riemann surface for the mapping as the segment of the characteristic $w_0 = w_0(t_0)$; this means that the segment *1—2* does not contain a cusp, and at every point is convex on the side of the region lying below the characteristic *1—2* in the tw plane (Fig. 5). From point *2* we draw the tangent to the characteristic *1—2* at point *2* to its intersection with the auxiliary curve $W = W(t)$ at point *3*; here $t_3 > t_1$. From the construction of the triangle *1—2—3* and consideration of the direction of convexity of the characteristics $w_0 = w_0(t_0)$ and $1-2$ follows the inequality, opposite to (8)

$$\left(\frac{dw}{dt}\right)_1 > \left(\frac{dw_0}{dt_0}\right)_2 = \left(\frac{dW}{dt}\right)_3 > \left(\frac{dW}{dt}\right)_1$$

Here the inequality sign cannot be replaced by an equality sign, because $y \neq \text{const}$ along the characteristic. Thus the case shown in Fig. 5 cannot exist.

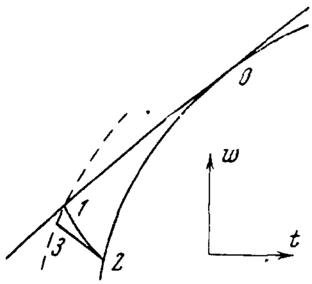

Fig. 5

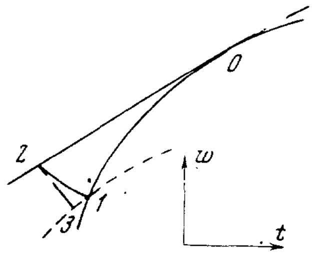

Fig. 6

We now consider the second case (Fig. 6). In the strip $0 < t < t_0$ we draw the auxiliary curve below the characteristic $w_0 = w_0(t_0)$. We denote by the numeral 1 the point of intersection of the curve $W = W(t)$ with the branch line; this point exists because $|t - t_0| > 0$. As in the previous case we first obtain

$$0 < \left(\frac{dw}{dt}\right)_1 \leqslant \left(\frac{dW}{dt}\right)_1 \tag{9}$$

Just as in the previous case we draw the segments $1-2$ and $2-3$; the segment $1-2$ is convex on the side of the region under the characteristic $1-2$, and $t_3 < t_1$.

We denote by ε the length of the segment $2-3$. For $\varepsilon \to 0$ we have the estimates

$$\rho(1,\ 2) = O(\varepsilon), \qquad \rho(1,\ 3) = O(\varepsilon^2)$$

where ρ is the length of the corresponding segment.

From the construction of the triangle $1-2-3$ we obtain

$$\left(\frac{dw}{dt}\right)_1 = \left(\frac{dw_0}{dt_0}\right)_2 - O(\varepsilon) = \left(\frac{dW}{dt}\right)_3 - O(\varepsilon) = \left(\frac{dW}{dt}\right)_1 - O(\varepsilon) + O(\varepsilon^2)$$

Here $O(\varepsilon)$ and $O(\varepsilon^2)$ are positive quantities of order ε and ε^2. For sufficiently small values of ε we obtain the inequality, opposite to (9)

$$(dw/dt)_1 < (dW/dt)_1$$

Thus in the second case the formation of an envelope of characteristics in the tw plane cannot exist.

13. The property obtained in Sect. 12 permits an easy extension to the case of axisymmetric transonic flow in zone 1a of the theorem on the breakdown of plane continuous supersonic flow in zone 1 due to straightening of a section of arbitrary length of the contour bounding this zone [2]. In this extension the role of straightening is played by w-straightening of the body contour.

In fact, according to Sect. 12 we find that if the shape of the body contour bounding zone 1a is given in the form of the curve $w = w(t)$, then

$$|dw/dt| > y$$

Since a w-straightened contour violates this bound (on it $dw/dt = 0$), either a shock wave arises or the location of zone 1a changes so that the w-straightened part no longer belongs to its boundary. In the case of uniqueness of the solution in zone 1a and its continuous dependence on changes in the boundary conditions, the second possibility must be rejected. For arbitrarily small length of the w-straightened part of the contour, the length of the resulting shock wave must be a quantity of the same order of smallness.

In the case of zone 1b, extension of the theorem of [2] can be obtained for that subregion lying in a sufficiently small neighborhood of the sonic point on the body whose mapping in the tw plane is single-sheeted (up to the carrying out of w-straightening of a part of the contour in this vicinity). We carry out the proof here analogously to [8].

Integrating Eq. (7) along the characteristics, we obtain

$$w(t) = \int_0^t y(t, w(t))\,dt + \lambda, \qquad w(t) = -\int_0^t y(t, w(t))\,dt + \mu$$

Here λ and μ are constants equal to the values of w at the points of intersection of the characteristics with the sonic line. Differentiating these equations with respect to λ and μ, we obtain

$$w_\lambda = 1 + yt_\lambda + \int_0^t y_w(t, w(t))\,w_\lambda dt, \qquad w_\lambda = -yt_\lambda$$

$$w_\mu = 1 - yt_\mu - \int_0^t y_w(t, w(t))\,w_\mu dt, \qquad w_\mu = yt_\mu$$

Hence

$$2w_\lambda = 1 + \int_0^t y_w w_\lambda dt, \qquad 2w_\mu = 1 - \int_0^t y_w w_\mu dt \tag{10}$$

We show that w-straightening of the contour in a sufficiently small vicinity of the sonic point may not bring only violation of single-sheetedness of the mapping of zone 1b in the tw plane. In fact, if the contour intersects a branch line, then at that point the derivative in the direction of one of the characteristics changes sign; let this be $\partial w / \partial s_1$. The derivative $\partial w / \partial s_1$ can be represented in the form

$$\frac{\partial w}{\partial s_1} = \frac{\partial w}{\partial \mu} h_1 = \frac{\partial w}{\partial \mu} \frac{\lambda_x \mu_y - \lambda_y \mu_x}{\sqrt{\lambda_x^2 + \lambda_y^2}}$$

The Lamé coefficient h_1 does not vanish for $0 < t < \infty$, because this is only possible either at points of tangency of characteristics or at points at which simultaneously $\mu_x = 0$ and $\mu_y = 0$. The latter is impossible at isolated points, since there are not multiple singular points on a characteristic, and the equality $\mu_x = \mu_y = 0$ on some line means that this line is a characteristic and that it is a branch line. As shown in Sect. 4, this is possible only if a weak discontinuity in the first derivative of a velocity component propagates along the characteristic; it is always possible to carry out w-straightening so that the curvature of the contour is left continuous. Thus it is found that at the point of intersection of a branch line with the contour the derivative w_μ vanishes.

It follows from Eq. (10) that at the sonic point of the contour $w_\mu = 1/2$, so that on a smooth contour there exists a neighborhood of the sonic point in which $w_\mu \geqslant \delta > 0$. With w-straightening of part of the contour in this neighborhood, the derivative w_μ

changes by not less than δ, and the magnitude of this change (which can arise only on account of a change in the value of the integral in (10)) depends not upon the length of the w-straightened part, but only on its location in the vicinity of the sonic point.

Thus if the w-straightened contour in the indicated neighborhood is not accompanied by formation of a shock wave, then either the uniqueness of the solution is violated or its continuous dependence on changes in boundary conditions.

14. For investigating plane potential flows close to prescribed ones, the derivation of "equations of variation" in the variables of the hodograph for the unperturbed flow was given in [9]. These equations have repeatedly been used subsequently for investigating the fundamental problems of transonic flow theory by the ABC method of Friedrichs.

We derive analogous equations for the study of axisymmetric transonic flow. Taking the original equations in the form (1), we introduce the stream function ψ and potential φ by the equations

$$\psi_x = w(k+1)^{-1}, \quad \psi_y = y\frac{u^2}{2}(k+1)^{-1} - y, \quad \varphi_x = 1 + u(k+1)^{-1}, \quad \varphi_y = w[y(k+1)]^{-1}$$

Varying these equations at a fixed point of the xy plane and denoting the variations by primes, we obtain

$$(k+1)\varphi_x' = u', \quad y(k+1)\varphi_y' = w', \quad (k+1)\psi_x' = w', \quad (k+1)\psi_y' = yuu'$$

Hence

$$yu\varphi_x' = \psi_y', \qquad \psi_x' = y\varphi_y' \tag{11}$$

We take as independent variables those of the hodograph of the unperturbed flow u, w. Substituting the equations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u}u_x + \frac{\partial}{\partial w}w_x, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial u}u_y + \frac{\partial}{\partial w}w_y$$

into Eqs. (11), we obtain

$$\psi_u'u_y + \psi_w'w_y = yu(\varphi_u'u_x + \varphi_w'w_x); \quad \psi_u'u_x + \psi_w'w_x = y(\varphi_u'u_y + \varphi_w'w_y)$$

Combining these equations we obtain, with the use of (11),

$$\psi_w' = \varphi_u', \qquad \psi_u' = y^2u\varphi_w'$$

Eliminating φ' or ψ' from this system, we obtain respectively

$$\psi_{ww}' = \left(\frac{\psi_u'}{y^2u}\right)_u, \qquad \varphi_{uu}' = u(y^2\varphi_w')_u$$

It is interesting to note that these equations differ from the corresponding equations of plane transonic flow only by the positive coefficient y^2, which is determined from the basic solution.

The fact that y^2 depends on both u and w makes impossible a direct carry-over of the results obtained in the theory of plane flow.

For an investigation of uniqueness of the solution of the problem "in the small", the contours of the body remain unchanged. Therefore the corresponding boundary values are written, as in the plane case, in the form $\psi_i = C_i$ on each of the curves $w_i = w_i(u)$ that represents the contour of the body in the basic flow.

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APPROXIMATION OF OPTIMAL GAME STRATEGIES BY CONTINUOUS FUNCTIONS

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We consider three typical game problems in conflict-control systems. We establish that in the regular case the optimal methods of control can be approximated by continuous strategies so as to achieve an effect as near optimal as desired (from the viewpoint of the pursuer or the pursued).

1. Let us consider the motion $x(t) = \{x_i(t)\}$, $(i = 1, \ldots, n)$ described by the vector differential equation

$$dx / dt = A(t)x + B(t)u - C(t)v + f(t) \tag{1.1}$$

Here $A(t)$, $B(t)$, $C(t)$ are matrices of dimensions $n \times n$, $n \times r$, $n \times s$ respectively; $f(t)$ is an n-dimensional perturbation vector; u and v are control vectors of dimensions